GAUSSIAN APPROXIMATION FOR NONMARKOVIAN STOCHASTIC NETWORKS IN HEAVY TRAFFIC

H. V. Livinska, E. O. Lebedev

National Taras Shevchenko University of Kyiv
Kyiv, Ukraine
E-mail: livinskaav@gmail.com, leb@unicyb.kiev.ua

In the paper multichannel stochastic networks of $[M_t \mid GI \mid \infty]^r$-type are considered. A non-homogeneous Poisson input flow of calls arrives at each node. Convergence of the service process in the uniform topology in heavy traffic is proved. Limit non-Markov Gaussian processes with the correlation characteristics written out in an explicit form via the model parameters are constructed.

Keywords: multichannel stochastic network, heavy traffic regime, non-homogeneous Poisson input flow, Gaussian approximation.

In the analysis of data transmission networks, computer networks and modern communication networks researchers often use models of stochastic systems and networks [1–3]. Their structure is determined by the probabilistic characteristics of the input information flows, data processing algorithms. Typically, the information processing in stochastic networks are high-dimensional vectors with interconnected components and complex system of stochastic relations defining the process. Therefore, the asymptotic methods of research are particularly effective in the study of stochastic networks. They allow you to see the regularities which underlie the functioning process of stochastic networks of the given type. Approximative process and the associated functionals allow to calculate the basic parameters of the process of information processing in order to solve optimization problems.

The basic mathematical model under consideration in the paper is a stochastic network consisting of "*r*" processing nodes (Fig. 1).

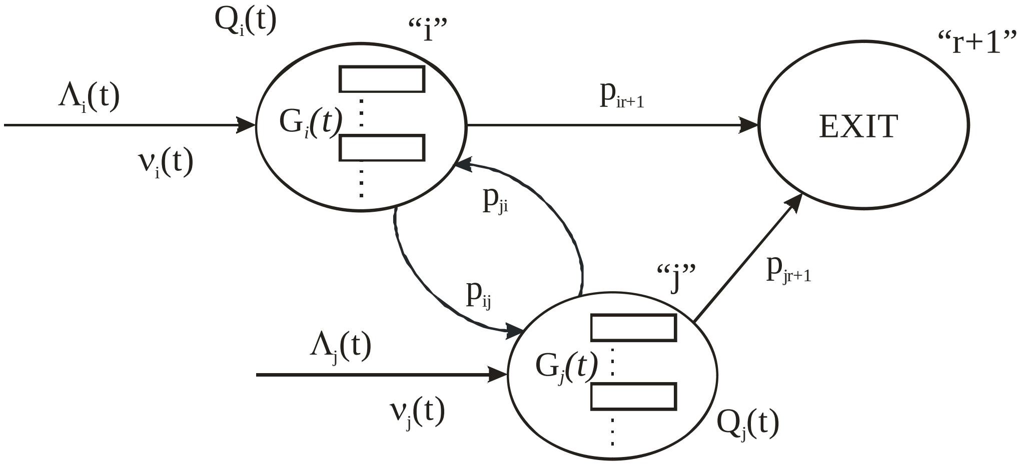

Fig. 1. Multichannel stochastic networks of $[M_t \mid GI \mid \infty]^r$ -type

We consider a non-homogeneous Poisson input flow of information packets $\nu_i(t)$ arriving at the *i*-th node, $i = 1, 2, ..., r$, considering $\Lambda_i(t)$ as the leading function of $\nu_i(t)$. Each of "*r*" nodes operates as a multi-channel stochastic system. If an information packet arrives at such a system then its service immediately begins. The distribution of the processing time depends on the node number. Let us denote its distribution function by $G_i(t)$, $i = 1, 2, ..., r$. After processing in the *i*-th node the packet arrives in the j-th node with probability p_{ij} and leaves the network with probability $p_{ir+1} = 1 - \sum_{j=1}^{r} p_{ij}$. $P = \left\| p_{ij} \right\|_1^r$ is a switching matrix of the network. An additional node numbered "$r+1$" is interpreted as an "output" from the network.

We will assume $G_i(0+) = 0$, $i = 1, 2, ..., r$, which is always realized in practice. According to the system notation, which is adopted in the theory of stochastic networks, such a model will be marked by the symbol $[M_t \,|\, GI \,|\, \infty]^r$.

The processing of information packets in a network of this type is defined as an *r*-dimensional stochastic process $Q'(t) = (Q_1(t), ..., Q_r(t))$, where $Q_i(t)$, $i = 1, 2, ...r$, is the number of packets in the *i*-th node at *t* moment of time.

The path of an information packet from the input moment of time in the $[M_t \,|\, GI \,|\, \infty]^r$-network through the *i*-th node until output of it can be described by a semi-Markov process $x^{(i)}(t) \in \{1, 2, ..., r, r+1\}$, which are defined by the following semi-Markov matrix $\left\| G_{ij}(t) \right\|_1^{r+1}$:

$$G_{ij}(t) = \begin{cases} p_{ij} G_i(t), \ i = 1,2,\ldots r; \ j = 1,2,\ldots,r,r+1, \\ \delta_{r+1j} G_{r+1}(t), \ i = r+1, \ j = 1,2,\ldots,r,r+1, \end{cases} \quad G_{r+1}(t) = \begin{cases} 0, & t < 1, \\ 1, & t \ge 1. \end{cases}$$

and by the initial conditions:

$$P\left\{ x^{(i)}(0) = i \right\} = \delta_{ij}, \quad j = 1, 2, ..., r, r+1;$$

a distribution function of the residence time in the initial state *i* which coincides with the $G_i(t)$.

Let us note that the state "$r+1$" for the process $x^{(i)}(t)$ is an absorbing state. The absorption in the state "$r+1$" is interpreted as output of packet from the network. Let us denote by $p^i{}_j(t) = P(x^{(i)}(t) = j)$ transitional probabilities of the semi-Markov process $x^{(i)}(t)$, $P(t) = \left\| p^i{}_j(t) \right\|_1^{r+1}$.

In order to have a further analysis it is also required such functions:

$$p^i_{j_1, j_2, ..., j_k}(t_1, t_2, ..., t_k) = P\left(x^{(i)}(t_1) = j_1, ..., x^{(i)}(t_k) = j_k \right),$$

$$i = 1, 2, ..., r, \ \ t_1 < t_2 < ... < t_k,$$

which are uniquely determined by a sequence of systems of Markov renewal integral equations:

$$p^i_{j_1}(t_1) = \sum_{m=1}^{r} \int_0^{t_1} dG_{im}(u) p^m_{j_1}(t_1 - u) + \delta_{ij_1}\left[1 - G_i(t_1)\right], \quad k = 1,$$

$$p^i_{j_1, j_2, ..., j_k}(t_1, t_2, ..., t_k) =$$

$$= \sum_{m=1}^{r} \int_0^{t_1} dG_{im}(u) p^m_{j_1, j_2, ..., j_k}(t_1 - u, t_2 - u, ..., t_k - u) +$$

$$+\sum_{\alpha=2}^{k}\left\{\delta_{ij_1}\ldots\delta_{ij_{\alpha-1}}\sum_{m=1}^{r}\int_0^{t_\alpha} dG_{im}(u)p^m_{j_\alpha,j_{\alpha+1},\ldots,j_k}(t_\alpha-u,t_{\alpha+1}-u,\ldots,t_k-u)\right\}+$$

$$+\delta_{ij_1}\ldots\delta_{ij_k}\left[1-G_i(t_k)\right],\qquad k>1. \tag{1}$$

We will study the process of information packets processing in a heavy traffic regime.

This means that $[M_t\,|\,GI\,|\,\infty]^r$-network parameters depend on n (a series number) so that the following conditions are satisfied:

Condition 1. The input flows depend on n so that in any finite interval $\left[0,T\right]$ we have

$$n^{-1}\Lambda_i^{(n)}(nt)\underset{n\to\infty}{\overset{U}{\Rightarrow}}\Lambda_i^{(0)}(t)\in C\left[0,T\right],\ \ i=1,2,\ldots,r,$$

where $C\left[0,T\right]$ is a set of continuous functions in the interval $\left[0,T\right]$, let us note that the symbol $\overset{U}{\Rightarrow}$ means convergence in the uniform topology.

Condition 2. $G_i^{(n)}(nt)\underset{n\to\infty}{\overset{d}{\Rightarrow}}G_i(t)$, $i=1,2,\ldots r$,

where the symbol $\overset{d}{\Rightarrow}$ means convergence by distribution.

Let us consider two cases that are important for applications when the Condition 1 is held. We temporarily assume that the Poisson flow $\nu_i(t)$ is regular: $\Lambda_i(t)=\int_0^t\lambda_i(u)du$, where $\lambda_i(u)$ is an instant value of the parameter.

a) If $\lambda_i(t)$ is a periodic function with period T_i then Condition 1 holds with $\Lambda_i^{(0)}(t)=\left(\int_0^{T_i}\lambda_i(u)du\right)t$.

b) If $\lim_{t\to\infty}\lambda_i(t)=\lambda_i>0$ then Condition 1 holds with $\Lambda_i^{(0)}(t)=\lambda_i t$.

In context of Conditions 1, 2 we consider a normalized process of data handling in the nodes of the $[M_t\,|\,GI\,|\,\infty]^r$-network, that is:

$$\xi^{(n)\prime}(t)=n^{-1/2}\left(Q^{(n)}(nt)-\int_0^{nt}[d\Lambda^{(n)}(\tau)]'P^{(n)}(nt-\tau)\right),$$

where

$$\left[d\Lambda^{(n)}(\tau)\right]'=\left(d\Lambda_1^{(n)}(\tau),\ldots,d\Lambda_r^{(n)}(\tau)\right),$$

$$P^{(n)}(t)=\left\|p_j^{i(n)}(t)\right\|_{i,j=1}^{r},$$

$$p_j^{i(n)}(t)=P\left(x^{(i,n)}(t)=j\right),$$

$x^{(i,n)}(t)$ is a semi-Markov process which is defined as $x^{(i)}(t)$ with replacement of the distribution functions from $G_i(t)$ to $G_i^{(n)}(t)$, $i=1,2,\ldots r$.

To describe limit behavior of the sequence of stochastic processes $\xi^{(n)}(t)$, $n\geq 1$, we introduce two independent Gaussian processes $\xi^{(i)\prime}(t)=(\xi_1^{(i)}(t),\ \ldots,\ \xi_r^{(i)}(t))$, $i=1,2$.

The processes $\xi^{(1)}(t)$ and $\xi^{(2)}(t)$ are determined by the null mean values and by correlation matrices:

$$R^{(1)}(t) = E\xi^{(1)}(t)\xi^{(1)\prime}(t) - E\xi^{(1)}(t)E\xi^{(1)\prime}(t) = \int_0^t P'(t-\tau)\Delta[d\Lambda^{(0)}(\tau)]P(t-\tau)\,,$$

$$R^{(1)}(s,t) = E\xi^{(1)}(s)\xi^{(1)\prime}(t) - E\xi^{(1)}(s)E\xi^{(1)\prime}(t) = \int_0^t P'(s-\tau)\Delta[d\Lambda^{(0)}(\tau)]P(t-\tau),$$

$$R^{(2)}(t) = \int_0^t [\Delta[(d\Lambda^{(0)}(\tau))'P(t-\tau)] - P'(t-\tau)\Delta[d\Lambda^{(0)}(\tau)]P(t-\tau)]\,,$$

$$R^{(2)}(s,t) = \sum_{m=1}^{r}\int_0^s \{[\Delta(p_m(s-\tau)) - p_m(s-\tau)p'_m(s-\tau)]\times$$
$$\times E^{(m)}(s-\tau,t-\tau)d\Lambda_m{}^{(0)}(\tau)\}\,,$$

where $s<t$, $\left(d\Lambda^{(0)}(\tau)\right)' = \left(d\Lambda_1^{(0)}(\tau),...,d\Lambda_r^{(0)}(\tau)\right)$, $p'_m(t) = \left(p_1^m(t),...,p_r^m(t)\right)$ is the m-th row of the matrix $P(t)$,

$$E^{(m)}(s,t) = \left\| {p_{ij}{}^{(m)}(s,t)} \Big/ {p_i^m(s)} \right\|_{i,j=1}^{r}\,,$$

$$p_{ij}^{(m)}(s,t) = P(x^{(m)}(s)=i, x^{(m)}(t)=j)\,,$$

and $\Delta(x) = \left\|\delta_{ij}x_i\right\|_{i,j=1}^{r}$ is a diagonal matrix for any vector $x' = \left(x_1,...,x_r\right)$.

For the normalized process of processing data packets $\xi^{(n)}(t),\ n\geq 1$, the following result holds.

Theorem 1. Let for the $[M_t\,|\,GI\,|\,\infty]^r$-network Conditions 1, 2 take place and at the initial moment of time $t=0$ the network is empty: $Q^{(n)\prime}(0) = (0,...,0)$. Then for any finite interval $\left[0,T\right]$ the sequence of stochastic processes $\xi^{(n)}(t),\ n\geq 1$ weakly converges in the uniform topology to $\xi^{(1)}(t)+\xi^{(2)}(t)$.

Note that a part $\xi^{(1)}(t)$ of the limit process is associated with fluctuations of the input flows and $\xi^{(2)}(t)$ with fluctuations of the service times.

A proof of Theorem 1 follows from some auxiliary results.

Lemma 1. Let $\nu^{(n)}(t),\ n\geq 1$ be a non-homogeneous Poisson process with the leading function $\Lambda(t) = \Lambda^{(n)}(t)$ dependent on the series number "n" for which Condition 1 is true. Then for any finite interval $\left[0,T\right]$ the sequence of stochastic processes

$$W^{(n)}(t) = n^{-1/2}\left(\nu^{(n)}(nt) - \Lambda^{(n)}(nt)\right)$$

converges in the uniform topology to the Wiener process $W^{(0)}(t)$ with $EW^{(0)}(t)=0$ and $VarW^{(0)}(t) = \Lambda^{(0)}(t)$.

Proof. Convergence of finite-dimensional distributions of the process $W^{(n)}(t)$ to $W^{(0)}(t)$ follows from the fact that for any natural number N and time moments $0 < t_1 < ... < t_N$ a characteristic function of a joint distribution of $\nu^{(n)}(t_1), ..., \nu^{(n)}(t_N)$, is equal to

$$E\exp\left\{ i\sum_{k=1}^{N} s(k)\nu^{(n)}(t_k) \right\} =$$

$$= \prod_{k=0}^{N-1} \exp\left\{ \left[\Lambda^{(n)}(t_{k+1}) - \Lambda^{(n)}(t_k) \right] \left[\exp\left\{ i\sum_{m=k+1}^{N} s(m) \right\} - 1 \right] \right\},$$

where $\left(s(1), ..., s(N)\right) \in R_N$, $t_0 = 0$.

Now in order to prove the convergence in the uniform topology it's sufficient to check the following condition (see [4]):

$$\lim_{h\to 0} \overline{\lim_{n\to\infty}} \sup_{|t_1 - t_2| \le h} P\left\{ \left| W^{(n)}(t_2) - W^{(n)}(t_1) \right| > \varepsilon \right\} = 0 \qquad (2)$$

for any $\varepsilon > 0$.

Based on the Chebyshev inequality we have the upper estimate:

$$P\left\{ \left| W^{(n)}(t_2) - W^{(n)}(t_1) \right| > \varepsilon \right\} =$$

$$= P\left\{ \left| \nu^{(n)}(nt_2) - \nu^{(n)}(nt_1)\left(\Lambda^{(n)}(nt_2) - \Lambda^{(n)}(nt_1) \right) \right| > \varepsilon\sqrt{n} \right\} \le$$

$$\le \varepsilon^{-2} \left| n^{-1}\Lambda^{(n)}(nt_2) - n^{-1}\Lambda^{(n)}(nt_1) \right|. \qquad (3)$$

On the other side Condition 1 implies that

$$\lim_{h\to 0} \overline{\lim_{n\to\infty}} \sup_{|t_1 - t_2| \le h} \left| n^{-1}\Lambda^{(n)}(nt_2) - n^{-1}\Lambda^{(n)}(nt_1) \right| = 0. \qquad (4)$$

So, (2) follows from (3) and (4). The lemma is proved.

Hereafter we will denote as $W_i^{(0)}(t)$, $i = 1, 2, ..., r$, independent Wiener processes with $EW_i^{(0)}(t) = 0$ and $VarW_i^{(0)}(t) = \Lambda_i^{(0)}(t)$. If Condition 1 takes place then paths of $W_i^{(0)}(t)$ are continuous with probability 1, and the process $W^{(0)\prime}(t) = \left(W_1^{(0)}(t), ..., W_r^{(0)}(t) \right)$ approximates an external input flow $\nu^{(n)\prime}(t) = \left(\nu_1^{(n)}(t), ..., \nu_r^{(n)}(t) \right)$ in the $[M_t \,|\, GI \,|\, \infty]^r$-network.

For $W^{(0)\prime}(t) = \left(W_1^{(0)}(t), ..., W_r^{(0)}(t) \right)$ we will need the following result.

Lemma 2. Finite-dimensional distributions of $\int_0^t dW^{(0)\prime}(u)P(t-u)$ coincide with finite-dimensional distributions of the Gaussian process $\xi^{(1)}(t)$.

The proof of this result follows from the properties of a stochastic integral over a Gaussian process with independent increments.

Let us connect with the semi-Markov process $x^{(m)}(t)$ an r-dimensional process of indicator type $\chi^{(m)\prime}(t) = (\chi_1^{(m)}(t), ..., \chi_r^{(m)}(t))$, $t \ge 0$, as follows:

$$\chi^{(m)}(t) = \begin{cases} e_j, & x^{(m)}(t) = j, \ j = 1, 2, ..., r; \\ e_0, & x^{(m)}(t) = r+1; \end{cases} \qquad (5)$$

where e_i is an r-dimensional vector with the i-th component equal to 1, while others are null, e_0 is a null r-dimensional vector.

For an arbitrary natural N and $z'(k) = (z_1(k),..., z_r(k))$, $k = 1, 2,..., N$, $|z(k)| \le 1$, we will denote a joint generating function of the vectors $\chi^{(m)}(t_1),...,\chi^{(m)}(t_N)$, $0 < t_1 < ... < t_N$, as $\Phi^{(m)} = \Phi^{(m)}(t_1,...,t_N, z(1),..., z(N))$.

The function $\Phi^{(m)}$ can be represented as follows.

Lemma 3. Let $p^i_{j_1, j_2,..., j_k}(t_1, t_2,..., t_k)$ be functions defined by the system (1). Then for any $N = 1, 2,...$ and $0 < t_1 < ... < t_N$

$$\Phi^{(m)}\left(t_1,...,t_N, z(1),..., z(N)\right) = \bar{1} + \tag{6}$$

$$+\sum_{k=1}^{N} \sum_{i_1,...,i_k=1}^{r} p^m_{i_1,...,i_k}(t_1,...,t_k) z_{i_1}(1)...z_{i_{k-1}}(k-1)[z_{i_k}(k)-1].$$

The formula (6) can be obtained by a mathematical induction method on the parameter N.

A proof of the Theorem 1 can be obtained from the lemmas 1–3, if we represent the service process under a fixed input flow path as the sum of the indicators of the type (5) and carry out the necessary calculations.

In conclusion we note that the presented results extend models of Section 4.2 of the monograph [4] to the case of Poisson input flows with varying rates and service times with an arbitrary distribution functions. The markovian networks with a nonhomogeneous Poisson input flow and with exponentional distributed service times in heavy traffic were considered in [5–6].

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